The Local Hanf Number below $2^{\aleph_1}$

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This is a joint project with Dima Sinapova.

History

- In (January of) 1977 Shelah's published "A Two-Cardinal Theorem and a Combinatorial Theorem".
- The purpose of the paper is to prove that for *first-order theories* $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$.
- Shelah then conjectures that "if $\psi \in \mathcal{L}_{\omega_1,\omega}$ has a model of cardinality $\aleph_{\omega_1}$, then ψ has a model of size $2^{\aleph_0}$."
- If $2^{\aleph_0} \leq \aleph_{\omega_1}$, the result is trivial. So, rephrase:

Conjecture (Shelah)

In <u>all</u> models of ZFC+($2^{\aleph_0} > \aleph_{\omega_1}$), if $\psi \in \mathcal{L}_{\omega_1,\omega}$ has a model of cardinality $\aleph_{\omega_1}$, then ψ has a model of size $2^{\aleph_0}$.

History II

Conjecture (Shelah)

In all models of ZFC+($2^{\aleph_0} > \aleph_{\omega_1}$), if $\psi \in \mathcal{L}_{\omega_1,\omega}$ has a model of cardinality $\aleph_{\omega_1}$, then ψ has a model of size $2^{\aleph_0}$.

- (Assuming the conjecture is correct), Shelah calls $\aleph_{\omega_1}$ the *local Hanf number below* $2^{\aleph_0}$.
- The conjecture remains open as of today (43 years later).
- In 1999, Shelah published his result that the conjecture is consistent.
  - Start with a model V of ZFC+GCH.
  - Add enough Cohen reals so that $2^{\aleph_0} > \aleph_{\omega_1}$ in the extension.
  - In the extension the conjecture holds true.

Equivalent Formulation

The conjecture is equivalent to the following: For every $\psi \in \mathcal{L}_{\omega_1,\omega}$,

1. If κ is a cardinal in the interval $[\aleph_{\omega_1}, 2^{\aleph_0})$ and ψ has a model of size κ, then ψ has a model of size κ^+ (*you can not stop at a successor cardinal*) and
2. If (κ_i) is an increasing sequence in the interval $[\aleph_{\omega_1}, 2^{\aleph_0})$ and ψ has models in all cardinalities κ_i, then ψ has a model of size $\cup_i \kappa_i$.

(*you can not stop at a limit cardinal*)
This motivates the following definitions

Characterizable Cardinals

Definition

Let $\psi \in \mathcal{L}_{\omega_1,\omega}$.

- If ψ has models exactly in cardinalities $[\aleph_0, \kappa]$, then ψ *characterizes* κ.
- If κ is a limit cardinal and ψ has models exactly in cardinalities $[\aleph_0, \kappa)$, then ψ *limit characterizes* κ.

Hjorth's Theorem

Theorem (Hjorth)

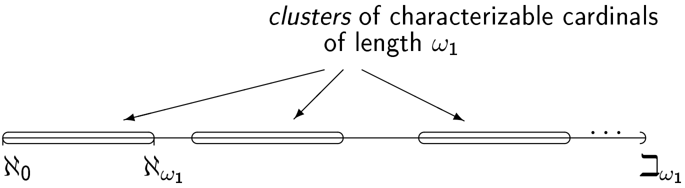
For all $\alpha < \omega_1$, there exists some ψ_α that characterizes $\aleph_\alpha$.

Corollary

If ψ characterizes some κ, then for every $\alpha < \omega_1$, there exists some ψ_α that characterizes $\kappa^{+\alpha}$(:the α^{th} successor of κ).

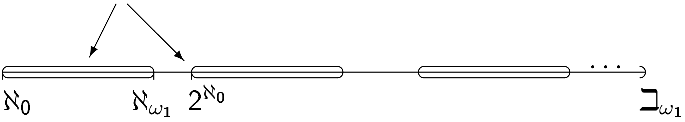
This says that characterizable cardinals come in "clusters" of length ω_1.

Clusters of Characterizable Cardinals



Shelah's Conjecture

Shelah's conjecture:
$2^{\aleph_0}$ is either below $\aleph_{\omega_1}$ or it is the first cardinal in the second cluster



Limit Characterizable Cardinals

- Limit characterizable cardinals have not been studied (yet!)
- Here is an easy example:
  - Let ϕ_n characterize $\aleph_n$.
  - Then $\bigvee_n \phi_n$ has models in cardinalities $[\aleph_0, \aleph_\omega)$.
  - I.e. $\bigvee_n \phi_n$ limit characterizes $\aleph_\omega$.

Open Questions

Open Questions

1. *Give examples of limit characterizable cardinals of uncountable cofinality. Can $2^{\aleph_0}$ be such an example?*
2. *Is it possible that some limit cardinal is characterizable, but not limit characterizable? Shelah's conjecture implies that if $2^{\aleph_0}$ is a limit cardinal, it is not limit characterizable.*
3. *Can we prove/disprove similar conjectures for higher cardinals? E.g. $2^{2^{\aleph_0}}$, $2^{\aleph_1}$, $\beth_\alpha, \ldots$*

Recent Developments

Theorem (Sinapova, S.)

There exists some $\psi \in \mathcal{L}_{\omega_1,\omega}$ so that the following are consistent

1. $2^{\aleph_0}$ *can be arbitrarily large and* ψ *characterizes* $2^{\aleph_0}$*;*
2. *CH (or* $\neg$*CH),* $2^{\aleph_1}$ *is a regular cardinal greater than* $\aleph_2$ *and* ψ *characterizes* $2^{\aleph_1}$*;*
3. $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ *and* ψ *characterizes* $\aleph_{\omega_1}$*; and*
4. *CH,* $2^{\aleph_1}$ *is the* $\left(2^{\aleph_1}\right)^{th}$*-weakly inaccessible and* ψ *limit characterizes* $2^{\aleph_1}$*. We need ZFC+ a Mahlo for this.*

The idea is that ψ *codes Kurepa trees. Turning on-off the existence of Kurepa trees, we get the corresponding consistency results.*

Comments

- This is the first non-trivial example of limit characterizing a cardinal.
- In fact, it is consistent that
  - ψ *characterizes* $2^{\aleph_1}$,
  - ψ characterizes some cardinal smaller than $2^{\aleph_1}$, and that
  - ψ *limit characterizes* $2^{\aleph_1}$.
- Ulrich and Shelah (in private communication) constructed a model of ZFC where
  - there is a local Hanf number below $\beth_2 = 2^{2^{\aleph_0}}$ and
  - the local Hanf number is no more than $\aleph_{\beth_1^{++}}$.
- In view of our result, there is not a good notion of a local Hanf number below $2^{2^{\aleph_0}}$, $2^{\aleph_1}$.

Idea of the Proof

- The sentence ψ codes a pseudo-tree (levels are not well-ordered, but they are linearly ordered) with countable levels.
- The height of the tree (i.e. the order type of the levels) is $\aleph_1$-like (every initial segment is countable).
- If the height is countable, the size of the model is bounded by $2^{\aleph_0}$.
- If the height is uncountable, we can embed ω_1 cofinally into the height.
- If the height is uncountable and there are more than $\aleph_1$-many branches, we can recover a Kurepa tree (not pseudo-tree).

Maximality Principles

Definition

1. Let Γ_κ be the class of κ-closed, stationary κ^+-linked, well met poset $\mathbb{P}$ with greatest lower bounds.
2. GMA_κ states that for every $\mathbb{P} \in \Gamma_\kappa$, and for every collection of less than 2^κ many dense sets there is a filter for $\mathbb{P}$ meeting them.
3. A sentence ϕ is Γ_κ-forceably necessary, if there is a Γ_κ forcing extension $V[G]$ such that ϕ holds true in all further Γ_κ forcing extensions $V[G][H]$ of $V[G]$.
4. For a regular κ, $SMP_n(\kappa)$ says $\kappa^{<\kappa} = \kappa$ and every Σ_n-sentence ϕ with parameters in $H(2^\kappa)$ which is Γ_κ-forceably necessary, ϕ is true in V.
5. $SMP(\kappa)$ is the statement that $SMP_n(\kappa)$ holds for all n.

Maximality Principles II

Theorem (Lücke)

1. *If κ satisfies $\kappa = \kappa^{<\kappa}$, then a model of SMP($\kappa$) can be forced starting from a Mahlo cardinal $\theta > \kappa$.*
2. *If V is a model of $SMP_2(\aleph_1)$, then the following hold true:*
   - *$GMA_{\aleph_1}$*
   - *CH*
   - *$2^{\aleph_1}$ is weakly inaccessible (in fact the $(2^{\aleph_1})^{th}$-weakly inaccessible).*
   - *every $\mathbf{\Sigma}^1_1$-subset of $\omega_1^{\omega_1}$ of cardinality 2^{ω_1} contains a perfect set.*

Corollary

In the above model, there is no Kurepa tree with $2^{\aleph_1}$ many branches, but for every $\aleph_2 \leq \lambda < 2^{\aleph_1}$, there is a Kurepa tree with λ-many branches.

Conclusion

Open Question

Does the above result generalizes to higher cardinalities?

If so, then there is no local Hanf number for any $\beth_\alpha$, $\alpha > 1$.

View #1: There is no local Hanf number at all. We just did not work hard enough to find a model of ZFC+$(2^{\aleph_0} > \aleph_{\omega_1})$ where Shelah's conjecture fails.
View #2: There is a local Hanf number below $2^{\aleph_0}$, but no higher. This indicates the specialness of $2^{\aleph_0}$.
E.g. there have been attempts (by Shelah and Baldwin-Laskowski) to prove the existence of $2^{\aleph_0}$-sized models using countable "approximations". Why does it take so long?

- Thank you!
- Copy of these slides will be posted at `http://souldatosresearch.wordpress.com/`
- Questions?