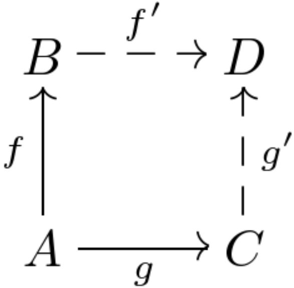
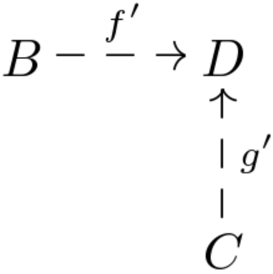
Amalgamation



Joint Embedding



Non- Absoluteness of Amalgamation and Joint-Embedding

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Spectra

Definition

Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$-sentence.

- ME(κ) (model-existence) is short for "ϕ has a model of size κ"
- AP(κ) (amalgamation) is short for "ME(κ) + the models of ϕ of size κ satisfy amalgamation"
- JEP(κ) (joint embedding) is short for "ME(κ) + the models of ϕ of size κ satisfy joint embedding"
- AP-Spec(ϕ) $= \{\kappa | AP(\kappa)\}$
- JEP-Spec(ϕ) $= \{\kappa | JEP(\kappa)\}$

Main Questions

1. *Is AP/JEP(κ) absolute (for transitive models of ZFC)?*
2. *(Baldwin) Is it possible for AP/JEP to hold-fail-hold-fail-... infinitely often?*

Absoluteness of Model Existence

By Shoenfield's absoluteness, $\mathrm{ME}(\aleph_0)$ is absolute.

Theorem (S.Friedman, Hyttinen, Koerwien)

1. *Model-existence in* $\aleph_1$ *is* **absolute**.
2. *Model-existence in* $\aleph_\alpha$, $1 < \alpha$, *is* **not absolute**.

The second part of the theorem can be proved by manipulating the size of the continuum.

Theorem (Milovich, S.)

Assuming a Mahlo cardinal, model-existence in $\aleph_\alpha$, $1 < \alpha < \omega_1$, *is* **not absolute** *even for models of ZFC+GCH.*

Theorem (Grossberg-Shelah)

Model-existence in any cardinal $\geq \beth_{\omega_1}$ *is* **absolute**.

Based on the (non-) absoluteness results for model-existence, we have the following questions.

- Is JEP/AP($\aleph_0$) absolute? Yes, by Shoenfield's absoluteness
- Is JEP/AP($\aleph_1$) absolute? This is open.
- Is JEP/AP non-absolute for all $\aleph_\alpha$, $1 < \alpha$? Yes, by manipulating the continuum.
- Assuming GCH, is JEP/AP non-absolute for all $\aleph_\alpha$, $1 < \alpha < \omega_1$? Mainly open. Yes, for AP and $1 < \alpha < \omega$.
- Is JEP/AP(κ) absolute for $\kappa \geq \beth_{\omega_1}$? Open for AP. "No" for JEP and κ limit cardinal less than the first measurable. "Yes" for $\kappa \geq$ the first measurable.

Aronszajn Trees

Definition

- A κ-tree is a tree of height κ such that all levels have size less than κ.
- A κ-Aronszajn tree is a κ-tree with no branch of length κ.
- A κ^+-tree is *special* if it is the union of κ-many of its antichains.
- $=^*$ means equality of sets modulo a finite set.
- A tree of functions is *coherent* if for every s, t with $dom(s) = dom(t)$, $s =^* t$.

A drawback in working with $\mathcal{L}_{\omega_1,\omega}$ is that we can not characterize well-orderings.
So, instead of working with (well-founded) trees, we have to work with *pseudo-trees*.

Definition

A *pseudotree* is a partial ordered set T such that each set of the form $\downarrow x = \{y | y <_T x\}$ is linearly ordered.

Theorem (Milovich, S.)

Given $1 \leq \alpha < \omega_1$, there is an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α satisfying the following.

1. *There is no model of ϕ_α of size greater than $\aleph_\alpha$.*
2. *If there is a coherent special $\aleph_\beta$-Aronszajn tree for each $\beta < \alpha$, then ϕ_α has a model of size $\aleph_\alpha$.*
3. *After collapsing a Mahlo to $\aleph_2$, ϕ_α has no model of size $\aleph_2$.*

Consider the collection of all models of ϕ_α with the *substructure* relation.

Theorem (Milovich, S.)

1. *Let $2 \leq \alpha < \omega$. If there are models of ϕ_α of size $\aleph_\alpha$, then AP-Spec(ϕ_α) $= \{\aleph_\alpha\}$. Otherwise AP-Spec(ϕ_α) is empty.*
2. *Let $\omega \leq \alpha < \omega_1$. The amalgamation spectrum of ϕ_α is empty.*

Corollary

*The following statements are **not absolute** for transitive models of ZFC+GCH.*

(a) *AP-Spec(ϕ) is empty.*

(b) *For finite $n \geq 2$, AP($\aleph_n$).*

The question for $\aleph_1$-amalgamation remains open.

Theorem (W.Boney, S.)

Let μ denote be the first measurable. There exists some Abstract Elementary Class $(\boldsymbol{K}, \prec)$ where $\boldsymbol{K}$ is the collection of all models of a certain $\mathcal{L}_{\omega_1,\omega}$-sentence and $A \prec B$ if $A \subset B$ and A, B "agree on finite sets" (low level complexity formula(s)), such that

1. *JEP($\aleph_0$) holds.*
2. *JEP(λ) fails for all $\aleph_1 \leq \lambda < \beth_\omega$.*
3. *If $\kappa < \mu$ and κ is a strong limit , then*
   - (i) *JEP(κ) holds, but*
   - (ii) *JEP(λ) fails, for all $\kappa^{<\kappa} \leq \lambda < \beth_\omega(\kappa)$.*
4. *If $\kappa \geq \mu$, then JEP(κ) holds.*

This is the first example where JEP holds-fails-holds-fails-... infinitely often.

Theorem (W.Boney, S.)

Assume GCH. Given a club C on the first measurable μ, there is a generic extension $V[G]$ that preserves cardinalities and cofinalities, μ remains a measurable cardinal, and $\boldsymbol{K}$ satisfies JEP(κ) iff $\kappa \in \lim C$ or $\kappa \geq \mu$.

Corollary

Let μ be the first measurable and let κ be a limit cardinal less than μ. Then JEP(κ) is not absolute.

Baldwin-Shelah proved that if $\kappa \geq \mu$, then JEP(κ) always holds and therefore, it is absolute.

Kurepa Trees

Definition

A Kurepa tree is an $\aleph_1$-tree with more than $\aleph_1$ many branches.

Theorem (Sinapova,S.)

There exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ that characterizes the maximum of $2^{\aleph_0}$ and $\mathcal{B}$.

$$\mathcal{B} = \sup\{\kappa | \text{there exists a Kurepa tree with } \kappa \text{ many branches}\}.$$

Note: $\mathcal{B} \leq 2^{\aleph_1}$.

Manipulating the size of Kurepa trees we can produce a variety of consistency result.

Theorem (Sinapova, S.)

There is an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ for which it is consistent that

- *AP-Spec*$(\phi) = [\aleph_1, 2^{\aleph_0}]$;
- *CH (or* $\neg$*CH) + "*$2^{\aleph_1}$ *is a regular cardinal greater than* $\aleph_2$*" + "AP-Spec*$(\phi) = [\aleph_1, 2^{\aleph_1}]$*";*
- $2^{\aleph_0} < \aleph_{\omega_1}$ *+ "AP-Spec*$(\phi) = [\aleph_1, \aleph_{\omega_1}]$*; and*
- $2^{\aleph_0} < 2^{\aleph_1}$ *+ "*$2^{\aleph_1}$ *is weakly inaccessible" + "AP-Spec*$(\phi) = [\aleph_1, 2^{\aleph_1})$*.*

Corollary

(a) *Let $\aleph_2 \leq \kappa \leq 2^{\aleph_1}$ and κ is a regular cardinal. Then AP(κ) is not absolute for models of ZFC+CH.*

(b) *It is not absolute for models of ZFC+GCH that AP-Spec(ϕ) is right-closed/open.*

Open Questions

- *Given a subset X of the cardinals, is there some ϕ, X=AP-Spec(ϕ)? Same question, but X=JEP-Spec(ϕ).*
- *Which specific subsets of the cardinals are not AP/JEP spectra?*
- *Are there two transitive models of ZFC $V \subset W$ and some $\phi \in (\mathcal{L}_{\omega_1,\omega})^V$ such that V, W agree on "ϕ has models of size $\aleph_2$", but disagree on "the models of ϕ satisfy AP($\aleph_2$)"?*

References

- Thank you!
- Copy of these slides can be found at `http://souldatosresearch.wordpress.com/`
- Questions?